MATH 1104 LINEAR ALGEBRA

LECTURE NOTES

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(These Lecture Notes replace neither the Text Book nor the Lectures)

Part 3

- Subspaces of R^n.
- Dimension and Rank.
- Determinants.
- Properties of Determinants.
- Cramer's Rule.
- Adjoint of a Matrix and Inverse via Adjoint.
- Determinants as Area and Volume.

SUBSPACES OF R^n

$$X \in R^n \Longleftrightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ or } X = (x_1,\ x_2,\ ...,\ x_n).$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

$$c \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}, \text{ where } c \in R.$$

Definition: A subset H of R^n is called a subspace if it has the following three properties:
(i) The zero vector is in H,
(ii) For each u and v in H, $u + v$ is also in H,
(iii) For each u in H and each scalar c, cu is also in H.

Example: $\text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is a line passing through the origin and the point $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and it is a subspace of R^2.

Remarks:

- If a line does not pass through the origin, it is not a subspace.
- R^n is a subspace of R^n.
  $\{0\}$ is a subspace of R^n.
- If $v_1, v_2, ..., v_k$ are vectors in R^n, then $\text{Span}\{v_1, v_2, ..., v_k\}$ is a subspace of R^n.
  This is called the subspace spanned by $v_1, v_2, ..., v_k$.

Example: Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Show that $H =$Span$\{e_1, e_2\}$ is a subspace of R^3.
Solution:

(i) $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Longrightarrow 0 \in H.$

(ii) Let $u = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in H$ and $v = c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in H$. Then

$$u + v = (a + c) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (b + d) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in H.$$

(iii) $cu = ca \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + cb \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in H.$

So, by (i), (ii) and (iii), H is a subspace.

Example: Determine which of the following sets are subspaces of R^3.

(i) $H_1 = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \middle| \, a \in R \right\}.$

(ii) $H_2 = \left\{ \begin{bmatrix} a \\ 1 \\ 1 \end{bmatrix} \middle| \, a \in R \right\}.$

(iii) $H_3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, b = a + c \right\}.$

(iv) $H_4 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, b = a + c + 1 \right\}.$

Solution: **(i)** Take $a = 0$. Then, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H_1$.

$$u + v = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \in H_1.$$

$$cu = c\begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} ca_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \in H_1.$$

Thus, H_1 is a subspace.

Another way to do (i):

$$u \in H_1 \Longrightarrow u = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow H_1 = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}, \text{ which is a subspace.}$$

(ii) $$\begin{bmatrix} b \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} c \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b + c \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ 2 \\ 2 \end{bmatrix} \notin H \Longrightarrow H_2 \text{ is not a subspace.}$$

(iii) $$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ a + c \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Longrightarrow H_3 = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

Thus, H_3 is a subspace.

(iv) $$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ a + c + 1 \\ c \end{bmatrix} \Longleftrightarrow \begin{cases} a = 0 \\ 1 = 0 \\ c = 0 \end{cases}, \text{ which is a contradiction.}$$

So, zero vector is not in $H_4 \Longrightarrow H_4$ is not a subspace.

Example: Let $H = \left\{ \begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix} \middle| \; t \in R \right\}$ and $W = \left\{ \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} \middle| \; s, \; t \in R \right\}$.

Show that H and W are subspaces of R^3 and R^4, respectively.
Solution:

$$u \in H \Longrightarrow u = \begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix} = t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \Longrightarrow H = \text{Span} \left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

$$w \in W \Longrightarrow w = \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \Longrightarrow W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\}.$$

Example: Let $W = \left\{ \begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix} \middle| \; a, \; b \in R \right\}$. Is W a subspace of R^3?

Solution:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix} \Longleftrightarrow \begin{array}{l} a = 1 \\ a = 6b \\ a = -2b \end{array},$$

which is not satisfied for any a and b. Hence $0 \notin W$, and so W is not a subspace.

Note: If u and v are two vectors in W, then $u + v$ is not in W either.

COLUMN SPACE AND NULL SPACE OF A MATRIX

Definition: Let A be an $m \times n$ matrix.

- The column space A, written as Col$A$, is the set of all linear combinations of the columns of A.

$$\boxed{\text{Col}A = \{b \in R^m : \ Ax = b \text{ for some } x \text{ in } R^n\}}$$

- The null space of A, written as Nul$A$, is the set of all solutions to the homogenous equation $Ax = 0$.

$$\boxed{\text{Nul}A = \{x \in R^n \ : \ Ax = 0\}}$$

Example: $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 7 & 6 \\ -4 & 6 & -2 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$.

a) Is b in the column space of A?
b) Find the column space of A.
c) Is $\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$ in the null space of A?

d) Find the null space of A.

Solution: a) $\left[\begin{array}{rrr|r} 1 & -3 & -4 & 3 \\ -3 & 7 & 6 & -5 \\ -4 & 6 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 5 & -3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

Since the equation $Ax = b$ consistent, $b \in$Col$A$.

$$Ax = b \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3-5t \\ -2-3t \\ t \end{bmatrix}, \ t \in R.$$

b) Since $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 7 & 6 \\ -4 & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$,

$$\text{Col}A = \text{Span}\left\{\begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \\ -2 \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix}\right\}.$$

Remark: $\text{Col}A \neq \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ since $\begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix} \notin \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

c) $A\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 7 & 6 \\ -4 & 6 & -2 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} \in \text{Nul}A.$

d)

$$\left[\begin{array}{rrr|r} 1 & -3 & -4 & 0 \\ -3 & 7 & 6 & 0 \\ -4 & 6 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$$Ax = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}, t \in R \Longrightarrow \text{NulA} = \text{Span}\left\{\begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}\right\}.$$

Theorem: Let A be an $m \times n$ matrix. Then,

(i) Col$A$ is a subspace of R^m.
(ii) Nul$A$ is a subspace of R^n.

Proof: **(i)** Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$ and

$$b_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, b_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, ..., b_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

For each i $(1 \leq i \leq n)$ $b_i \in R^m$ is a column of A and Col$A$ =Span$\{b_1, b_2, ..., b_n\}$. Thus Col$A$ is a subspace of R^m.

(ii) Let $0_{m\times 1}$ and $0_{n\times 1}$ be the zero vectors of R^m and R^n, respectively. Then

$$A_{m\times n}0_{n\times 1} = 0_{m\times 1} \Longrightarrow 0_{n\times 1} \in \text{ Nul}A.$$

Let u and v be in Nul$A$. Then, $Au = 0$ and $Av = 0$, and

$$A(u+v) = Au + Av = 0 + 0 = 0 \Longrightarrow u + v \in \text{ Nul}A.$$

For any scalar c and $u \in$ Nul$A$,

$$A(cu) = cAu = c\,0 = 0 \Longrightarrow cu \in \text{ Nul}A.$$

Thus, Nul$A$ is a subspace of R^n.

BASIS FOR A SUBSPACE

Definition: A basis for a subspace H of R^n is a linearly independent set (in H) which spans H.

Examples:

- $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis for R^2 (**standard basis**).
- $\left\{\begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$, $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$ are bases for R^2 as well.
- $\left\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$, $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$,

  $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$, $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\}$ are not bases for R^2.
- $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ is a basis for R^3 (**standard basis**).
- $\left\{\begin{bmatrix}-1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ is another basis for R^3.

- $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is not a basis for R^3.

- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is not a basis for R^3.

- $\{1, x, x^2, ..., x^n\}$ is a basis for P_n, the set of all polynomials of degree at most n.

- $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

  is a basis for $M_{2\times 2}$, the set of all 2×2 matrices.

Example: $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 7 & 6 \\ -4 & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

Col$A$ =Span$\left\{ \begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix} \right\}$ is a basis for Col$A$.

Nul$A$ =Span$\left\{ \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul$A$.

DIMENSION OF A SUBSPACE

Definition: The dimension of a nonzero subspace H, denoted by dim$H$, is the number of vectors in any basis for H.

The dimension of the subspace $\{0\}$ is 0.

The **rank** of a matrix A, denoted by rank$A$, is the dimension of the column space of A.

The Rank Theorem: If a matrix A has n columns, then

- $\boxed{\text{rank}A+\text{dim Nul}A = n}$ or
- $\boxed{\text{dim Col}A+\text{dim Nul}A = n}$.

Example: You are given that

$$A = \begin{bmatrix} 1 & -3 & 2 & 5 \\ -2 & 6 & 0 & -3 \\ 4 & -12 & -4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 5 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

(i) Find bases for Col$A$ and Nul$A$.
(ii) Find dim Col$A$ and dim Nul$A$.
(iii) Verify the Rank Theorem.

Solution: (i) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} \right\}$ is a basis for Col$A$.

$Ax = 0 \Longleftrightarrow Bx = 0$.
x_1, x_3 are basic variables and x_2, x_4 are free variables.

$$x_1 = 3x_2 - 2x_3 - 5x_4 = 3x_2 - 2(-7/4)x_4 - 5x_4 = 3x_2 - (3/2)x_4.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - (3/2)x_4 \\ x_2 \\ (-7/4)x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3/2 \\ 0 \\ -7/4 \\ 1 \end{bmatrix}.$$

A basis for Nul$A$ is $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 0 \\ -7/4 \\ 1 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ -7 \\ 4 \end{bmatrix} \right\}$.

(ii) rank$A$=dim Col$A = 2$, dim Nul$A = 2$.

(iii) rank$A +$ dim Nul$A = 4 = \#$ columns of A.

Example: You are given that

$$A = \begin{bmatrix} 3 & -5 & -1 & 4 & 4 \\ -2 & 4 & 2 & 7 & 8 \\ 5 & -9 & -3 & -3 & -4 \\ -2 & 6 & 6 & 5 & 9 \end{bmatrix} \sim \begin{bmatrix} 3 & -5 & -1 & 4 & 4 \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) Find bases for Col$A$ and Nul$A$.
(ii) Find dim Col$A$ and dim Nul$A$.
(iii) Verify the Rank Theorem.

Solution: (i) A basis for Col$A$ is $\left\{ \begin{bmatrix} 3 \\ -2 \\ 5 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ -3 \\ 5 \end{bmatrix} \right\}$.

A basis for Nul$A$: Need to solve $Ax = 0$.
x_1, x_2, x_4 are basic variables, x_3, x_5 are free variables.

$$x_4 = -x_5, \quad x_2 = -2x_3 - \frac{3}{2}x_5 \text{ and}$$

$$3x_1 = 5x_2 + x_3 - 4x_4 - 4x_5 = 5(-2x_3 - \frac{3}{2}x_5) + x_3 = -9x_3 - \frac{15}{2}x_5$$

$$\Longrightarrow x_1 = -3x_3 - \frac{5}{2}x_5.$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 - \frac{5}{2}x_5 \\ -2x_3 - \frac{3}{2}x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \left(\frac{1}{2}\right) \begin{bmatrix} -5 \\ -3 \\ 0 \\ -2 \\ 2 \end{bmatrix}.$$

Then, $\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ -2 \\ 2 \end{bmatrix} \right\}$ is a basis for Nul$A$.

(ii) rank$A$=dim Col$A = 3$, dim Nul$A = 2$.

(iii) rank$A +$ dim Nul$A = 5 = \#$ of columns of A.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 8 & 9 \end{bmatrix}$.

(i) Find bases for Col$A$ and Nul$A$.
(ii) Find dim Col$A$ and dim Nul$A$.
(iii) Verify the Rank Theorem.

Solution: Since $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, A is invertible.

Hence Nul$A = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ and Col$A = R^3$.

(ii) rank$A$=dim Col$A = 3$, dim Nul$A = 0$.

(iii) rank$A +$ dim Nul$A = 3 = \#$ of columns of A.

Remark: Let A be an $n \times n$ matrix. Then, the following conditions are equivalent.

(i) A is invertible.
(ii) dim Nul$A = 0$.
(iii) rank$A = n$.

Example: Find a basis for the subspace spanned by the vectors

$$\begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 9 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \\ -8 \\ -1 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 1 & -3 & -1 & 5 \\ -2 & 5 & 0 & -6 \\ -4 & 9 & -2 & -8 \\ 3 & -5 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & -7 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the subspace spanned by the given vectors is $\left\{ \begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 9 \\ -5 \end{bmatrix} \right\}$.

Remark:

$$\begin{bmatrix} -1 \\ 0 \\ -2 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 5 \\ 9 \\ -5 \end{bmatrix},$$

$$\begin{bmatrix} 5 \\ -6 \\ -8 \\ -1 \end{bmatrix} = -7 \begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix} - 4 \begin{bmatrix} -3 \\ 5 \\ 9 \\ -5 \end{bmatrix}.$$

COORDINATE VECTOR

Definition: Let H be a subspace, $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ be a basis for H, and $x \in H$. Then,

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

for some scalars $c_1,\ c_2, ..., c_n$.
The vector

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is called the coordinate vector of x relative to basis $\mathcal{B}$.

Examples: Consider the following bases for R^2.

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\},\ \mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$$

$$\mathcal{B}_3 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\},\ \mathcal{B}_4 = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Let $x = \begin{bmatrix} a \\ b \end{bmatrix} \in R^2$. Then

$$x = (-a+2b)\begin{bmatrix} 3 \\ 2 \end{bmatrix} + (2a-3b)\begin{bmatrix} 2 \\ 1 \end{bmatrix} \Longrightarrow [x]_{\mathcal{B}_1} = \begin{bmatrix} -a+2b \\ 2a-3b \end{bmatrix}.$$

Similarly,

$$[x]_{\mathcal{B}_2} = \begin{bmatrix} \dfrac{b}{2} \\ -a+\dfrac{b}{2} \end{bmatrix}, [x]_{\mathcal{B}_3} = \begin{bmatrix} \dfrac{a+b}{2} \\ \dfrac{a-b}{2} \end{bmatrix}, [x]_{\mathcal{B}_4} = \begin{bmatrix} \dfrac{-a+b}{2} \\ \dfrac{3a-b}{2} \end{bmatrix}.$$

Example: Let H be a subspace with a basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\} \text{ and } x = \begin{bmatrix} -9 \\ 7 \end{bmatrix}.$$

i) Find $[x]_{\mathcal{B}}$.
ii) Find y if $[y]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

Solution: i) We need to solve the equation

$$c_1\begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 \\ 7 \end{bmatrix}.$$

$$\left[\begin{array}{rr|r} 1 & -3 & -9 \\ -3 & 5 & 7 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & 6 \\ 0 & 1 & 5 \end{array}\right] \Longrightarrow \begin{cases} c_1 = 6, \\ c_2 = 5. \end{cases} \Longrightarrow [x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}.$$

ii)

$$y = 4\begin{bmatrix} 1 \\ -3 \end{bmatrix} + 6\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -14 \\ 18 \end{bmatrix}.$$

Example: Let H be a subspace with a basis

$$\mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ -6 \end{bmatrix} \right\} \text{ and } x = \begin{bmatrix} -8 \\ -1 \\ -3 \end{bmatrix}.$$

Find $[x]_{\mathcal{B}}$.

Solution:

$$\left[\begin{array}{rr|r} -3 & 7 & -8 \\ 1 & 5 & -1 \\ -4 & -6 & -3 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow [x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}.$$

Remark: $x \in R^3$ but $[x]_{\mathcal{B}} \in R^2$.

DETERMINANTS

Recall that for a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = ad - bc.$$

A is invertible $\Longleftrightarrow \det(A) \neq 0$.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Definition: Let $A = (a_{ij})$,

$(1 \leq i, j \leq n)$ be an $n \times n$ matrix, and let A_{ij} denote the submatrix formed by deleting the i th row and j th column of A. Then,

$$\begin{aligned} \det(A) &= (-1)^{1+1} a_{11} \det(A_{11}) \\ &\quad + (-1)^{1+2} a_{12} \det(A_{12}) \\ &\quad + \cdots \\ &\quad + (-1)^{1+n} a_{1n} \det(A_{1n}) \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}). \end{aligned}$$

This is called <u>cofactor expansion</u> along the first row.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$. Find $\det(A)$.

Solution: $A_{11} = \begin{bmatrix} 5 & 6 \\ -8 & 9 \end{bmatrix}$, $A_{12} = \begin{bmatrix} -4 & 6 \\ 7 & 9 \end{bmatrix}$, and $A_{13} = \begin{bmatrix} -4 & 5 \\ 7 & -8 \end{bmatrix}$.

$\det(A_{11}) = 45 - (-48) = 93$,

$\det(A_{12}) = -36 - 42 = -78$,

$\det(A_{13}) = 32 - 35 = -3$.

$$\begin{aligned} \det(A) &= (-1)^{1+1} a_{11} \det(A_{11}) + (-1)^{1+2} a_{12} \det(A_{12}) + (-1)^{1+3} a_{13} \det(A_{13}) \\ &= 1 \cdot 93 - 2 \cdot (-78) + 3 \cdot (-3) = 240. \end{aligned}$$

We can use the second row of A to calculate $\det(A)$:

$A_{21} = \begin{bmatrix} 2 & 3 \\ -8 & 9 \end{bmatrix}$, $A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$, $A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & -8 \end{bmatrix}$.

$\det(A_{21}) = 42$,

$\det(A_{22}) = -12$,

$\det(A_{23}) = -22$.

$$\begin{aligned} \det(A) &= (-1)^{2+1} a_{21} \det(A_{21}) + (-1)^{2+2} a_{22} \det(A_{22}) + (-1)^{2+3} a_{23} \det(A_{23}) \\ &= -(-4)(42) + 5(-12) - 6(-22) = 240. \end{aligned}$$

Let A be a 3×3 matrix. Then

$$\begin{aligned} \det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{22} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{11}a_{23}a_{32} + a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33}), \end{aligned}$$

which can be regarded as follows:
Use the first and second columns of A to form the fourt and fifth columns, respectively. Then, $\det A$ is obtained by adding the three downward diagonal products and subtracting the three upward diagonal products.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ -4 & 5 & 6 & -4 & 5 \\ 7 & -8 & 9 & 7 & -8 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 1\cdot 5\cdot 9 + 2\cdot 6\cdot 7 + 3\cdot(-4)\cdot(-8) \\ &\quad -\Big(7\cdot 5\cdot 3 + (-8)\cdot 6\cdot 1 + 9\cdot(-4)\cdot 2\Big) \\ &= 45 + 84 + 96 - (105 - 48 - 72) = 225 - (-15) = 240. \end{aligned}$$

Example: If $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 1 & 4 & 1 \\ -1 & 2 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix}$, what is $\det(A)$?

Solution:

$$\begin{aligned} \det(A_{11}) &= \begin{vmatrix} 1 & 4 & 1 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{vmatrix} \text{(by 1st column)} \\ &= 1\cdot\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix} = 1 - 2 + 0 = -1. \end{aligned}$$

$$\begin{aligned} \det(A_{12}) &= \begin{vmatrix} 2 & 4 & 1 \\ -1 & 1 & 0 \\ 1 & 3 & 1 \end{vmatrix} \text{(by 2nd row)} \\ &= -(-1)\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} + 1\cdot\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 0\cdot\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 1 + 1 + 0 = 2. \end{aligned}$$

We do not need to calculate $\det(A_{13})$ since $a_{13} = 0$.

$$\begin{aligned}\det(A_{14}) &= \begin{vmatrix} 2 & 1 & 4 \\ -1 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} \text{(by 3rd row)} \\ &= \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = -7 + 3 \cdot 5 = 8.\end{aligned}$$

$$\begin{aligned}\det(A) &= a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - a_{14}\det(A_{14}) \\ &= 1 \cdot (-1) - 2 \cdot 2 + 0 \cdot \det(A_{13}) - (-1) \cdot 8 = 3.\end{aligned}$$

Example: If $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ -5 & -8 & 4 & -3 \end{bmatrix}$, find $\det(A)$.

Solution: We can use cofactor expansion along the first row.

$$\det(A) = 4\ \det(A_{11}) = 4 \begin{vmatrix} -1 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & -3 \end{vmatrix} = 4(-1) \begin{vmatrix} 3 & 0 \\ 4 & -3 \end{vmatrix} = 36$$

- $\det(A) = 4 \cdot (-1) \cdot 3 \cdot (-3) = 36$.
- A square matrix is called upper triangular if all the entries below its main diagonal are zero.
- A square matrix is called lower triangular if all the entries above its main diagonal are zero.
- A matrix is called triangular if it is either upper triangular or lower triangular.
- The identity matrix is both upper triangular and lower triangular.
- If A is a triangular matrix, then $\det(A)$ is the product of the entries on its main diagonal.

Properties of Determinants

Let A and B be two $n \times n$ square matrices.

(i) $\det(A) = \det(A^T)$.

(ii) $\det(AB) = \det(A)\det(B)$

(iii) If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Proof of (iii): A is invertible $\Longleftrightarrow I = AA^{-1}$.

$$\det(I) = \det(AA^{-1}) \Longrightarrow 1 \overset{\text{(ii)}}{=} \det(A)\det(A^{-1}) \Longrightarrow \det(A^{-1}) = \frac{1}{\det(A)}.$$

Note: In general, $\det(A+B) \neq \det(A) + \det(B)$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ and $A+B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$.

$\det(A) + \det(B) = 1 + 8 = 9$ and $\det(A+B) = 23$.

So, $\det(A+B) \neq \det(A) + \det(B)$.

Homework: Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$. Show that

$$\det A + \det B = \det(A+B) \Longleftrightarrow a_1b_4 + a_4b_1 = a_3b_2 + a_2b_3.$$

Example: For $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & -1 \\ 17 & 6 \end{bmatrix}$,

$\det A + \det B = \det(A+B) = 57$.

The effect of row operations on determinants

Let A be an $n \times n$ square matrix.

- $A \overset{R_m \longleftrightarrow R_n}{\sim} B \Longrightarrow \det(B) = -\det(A)$.
- $A \overset{R_n' = kR_n}{\sim} B \Longrightarrow \det(B) = k\det(A)$.
- $A \overset{R_m' = R_m + kR_n}{\sim} B \Longrightarrow \det(B) = \det(A)$.

Note: Column operations have the same effects on determinants as row operations.

Remark 1: Let A be an $n \times n$ matrix.

- If A has two identical rows (or colums), then $\det A = 0$.
- If A has a zero-row (or zero column), then $\det A = 0$.

Remark 2: Let I_n be the $n \times n$ identity matrix and E be an $n \times n$ elementary matrix.

- $I_n \overset{R_p \longleftrightarrow R_s}{\sim} E \Longrightarrow \det(E) = -1$.
- $I_n \overset{R_p' = kR_p}{\sim} E \Longrightarrow \det(E) = k$.
- $I_n \overset{R_p' = R_p + kR_s}{\sim} E$, then $\det(E) = 1$.

Examples:

$$\begin{vmatrix} 1 & 3 & 2 & 3 \\ 4 & 4 & 5 & 4 \\ 5 & 1 & -3 & 1 \\ -7 & 2 & 8 & 2 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & 3 & 3 \\ 4 & 5 & 5 & -2 \\ 5 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 5,$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1 \quad , \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \qquad (R_2' = R_2 + 3R_4).$$

Example: Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 5 & 4 & 5 \\ -2 & 0 & 1 \end{bmatrix}$. Find $\det(A)$.

Solution:

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 2 & -1 \\ 5 & 4 & 5 \\ -2 & 0 & 1 \end{vmatrix} \begin{matrix} R_2' = R_2 - 5R_1 \\ R_3' = R_3 + 2R_1 \end{matrix} \\ &= \begin{vmatrix} 1 & 2 & -1 \\ 0 & -8 & 10 \\ 0 & 4 & -1 \end{vmatrix} = \begin{vmatrix} -8 & 10 \\ 4 & -1 \end{vmatrix} = -32. \end{aligned}$$

Remark: Let A be an $n \times n$ matrix.

i) A is invertible $\iff \det A \neq 0$

ii) $Ax = 0$ has nontrivial solutions $\iff \det A = 0$

Example: Let $\begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{bmatrix}$. Find $\det(A)$.

Solution:

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{vmatrix} \quad R_1' = R_1 + 2R_3 \\
&= \begin{vmatrix} 7 & 0 & 1 & 3 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{vmatrix} \text{(using 2nd column)} \\
&= -(-1)\begin{vmatrix} 7 & 1 & 3 \\ -1 & 2 & -2 \\ 2 & -1 & 2 \end{vmatrix} \quad \begin{matrix} R_2' = R_2 - 2R_1 \\ R_3' = R_3 + R_1 \end{matrix} \\
&= \begin{vmatrix} 7 & 1 & 3 \\ -15 & 0 & -8 \\ 9 & 0 & 5 \end{vmatrix} = -\begin{vmatrix} -15 & -8 \\ 9 & 5 \end{vmatrix} = 3
\end{aligned}$$

Example: If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -3$, then what is $\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix}$?

$$\begin{aligned}
\underbrace{\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix}}_{C_2'=C_2+C_1} &= \begin{vmatrix} a_1+b_1 & 2a_1 & c_1 \\ a_2+b_2 & 2a_2 & c_2 \\ a_3+b_3 & 2a_3 & c_3 \end{vmatrix} \\
&= 2\begin{vmatrix} a_1+b_1 & a_1 & c_1 \\ a_2+b_2 & a_2 & c_2 \\ a_3+b_3 & a_3 & c_3 \end{vmatrix} \quad C_1' = C_1 - C_2 \\
&= 2\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \quad C_1 \longleftrightarrow C_2 \\
&= -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-2)\cdot(-3) = 6.
\end{aligned}$$

Example: Let $A = \begin{bmatrix} a & b & c \\ u & v & w \\ x & y & z \end{bmatrix}$, $\det(A) = -6$. Evaluate the determinants of the following matrices.

(i) $A^{-1}A^TA$ **(ii)** $2A$ **(iii)** $\begin{bmatrix} a & b & 2a+c \\ u & v & 2u+w \\ x & y & 2x+z \end{bmatrix}$ **(iv)** $\begin{bmatrix} 3a & 3b & 3c \\ 4x & 4y & 4z \\ -u & -v & -w \end{bmatrix}$

Solution: **(i)**

$$\begin{aligned}
\det(A^{-1}A^TA) &= \det(A^{-1})\cdot\det(A^T)\cdot\det(A) \\
&= \frac{1}{\det(A)}\cdot\det(A)\cdot\det(A) = \det(A) = -6.
\end{aligned}$$

(ii) $\det(2A) = 2^3\det(A) = 8(-6) = -48.$

(iii) $\underbrace{\begin{vmatrix} a & b & 2a+c \\ u & v & 2u+w \\ x & y & 2x+z \end{vmatrix}}_{C_3'=C_3-2C_1} = \begin{vmatrix} a & b & c \\ u & v & w \\ x & y & z \end{vmatrix} = \det(A) = -6.$

$$
\begin{aligned}
\textbf{iv)} \quad & \begin{vmatrix} 3a & 3b & 3c \\ 4x & 4y & 4z \\ -u & -v & -w \end{vmatrix} \quad R_1' = \frac{1}{3}R_1 \\
= \; & 3\begin{vmatrix} a & b & c \\ 4x & 4y & 4z \\ -u & -v & -w \end{vmatrix} \quad R_2' = \frac{1}{4}R_2 \\
= \; & 3 \cdot 4\begin{vmatrix} a & b & c \\ x & y & z \\ -u & -v & -w \end{vmatrix} \quad R_3' = -R_3 \\
= \; & 12 \cdot (-1)\begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix} \quad R_2 \leftrightarrow R_3 \\
= \; & (-12)(-1)\begin{vmatrix} a & b & c \\ u & v & w \\ x & y & z \end{vmatrix} = 12 \cdot (-6) = -72.
\end{aligned}
$$

Example: If A and B are 5×5 matrices and $\det(-2A^T) = 128 = \det(A^3(B^T)^{-1})$, find $\det A$ and $\det B$.
Solution:

$\det(-2A^T) = (-2)^5 \det(A^T) = -32 \det A = 128 \Longrightarrow \det A = -128/32 = -4.$

$$
\begin{aligned}
\det(A^3(B^T)^{-1}) &= \det(A^3)\det((B^T)^{-1}) = (\det A)^3 \det(B^{-1})^T = (\det A)^3 \det(B^{-1}) \\
&= (-4)^3 \frac{1}{\det B} = 128 \Longrightarrow \det B = \frac{(-4)^3}{128} = -1/2.
\end{aligned}
$$

Example: Show A is an $n \times n$ matrix, then $\det(kA) = k^n \det(A)$.
Solution:

$$kA = (kI)A \det(kA) = \det((kI)A) = \det(kI) \cdot \det(A) = k^n \det(A).$$

$$\boxed{\det(kA) = k^n \det(A)}$$

Exercise: Suppose that

$$\det A = -3, \quad \det B = 2 \quad \text{and} \quad \det C = 5.$$

Compute $\det(A^2BC^{-1}B^T)$.

Exercise: Show that $\begin{bmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{bmatrix}$ has an inverse for any a, b and c.

CRAMER'S RULE

Let A be an <u>invertible $n \times n$ matrix</u>, $x = [x_1, x_2, ..., x_n]^T$. For any b in R^n,$Ax = b$ has a unique solution, and the entries of x are given by

$$x_i = \frac{\det A_i(b)}{\det A}, \ i = 1, 2, ..., n,$$

where $A_i(b)$ is the matrix obtained from A by replacing column i by the vector b.

Example: Let $A = \begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Solve $Ax = b$.

Solution: $A_1(b) = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}$, $A_2(b) = \begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix}$.

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{13}{13} = 1, \quad x_2 = \frac{\det A_2(b)}{\det A} = \frac{26}{13} = 2 \Longrightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Example: Use Cramer's rule to solve

$$\begin{aligned} x_1 + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8. \end{aligned}$$

Soultion: $A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix}$,

$$A_1(b) = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}, A_2(b) = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, A_3(b) = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}.$$

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix} \quad C_3' = C_3 - 2C_1 \\ &= \begin{vmatrix} 1 & 0 & 0 \\ -3 & 4 & 12 \\ -1 & -2 & 5 \end{vmatrix} = \begin{vmatrix} 4 & 12 \\ -2 & 5 \end{vmatrix} = 44 \end{aligned}$$

$$\begin{aligned} \det A_1(b) &= \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix} \quad C_1' = C_1 - 3C_3 \\ &= \begin{vmatrix} 0 & 0 & 2 \\ 12 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix} = 2\begin{vmatrix} 12 & 4 \\ -1 & -2 \end{vmatrix} = -40. \end{aligned}$$

Thus

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{-40}{44} = \frac{-10}{11}.$$

$$x_2 = \frac{\det A_2(b)}{\det A} = \frac{72}{44} = \frac{18}{11}.$$

$$x_3 = \frac{\det A_3(b)}{\det A} = \frac{152}{44} = \frac{38}{11}.$$

Example: Use Cramer's rule to solve for x without solving for y, z, and w.

$$\begin{aligned} -y+z+3w &= 1 \\ x+2y-z+w &= 2 \\ 3z+3w &= 0 \\ y+8z &= 1 \end{aligned}$$

Solution:

$$Ax=b \Longrightarrow \begin{bmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad A_1(b) = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 8 & 0 \end{bmatrix}.$$

$$\begin{aligned} \det A &= \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} = -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix} \; R_3' = R_3 + R_1 \\ &= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} = -(-1)\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} = -18. \end{aligned}$$

$$\begin{aligned} \det A_1(b) &= \begin{vmatrix} 1 & -1 & 1 & 3 \\ 2 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 8 & 0 \end{vmatrix} \quad C_3' = C_3 - C_4 \\ &= \begin{vmatrix} 1 & -1 & -2 & 3 \\ 2 & 2 & -2 & 1 \\ 0 & 0 & 0 & 3 \\ 1 & 1 & 8 & 0 \end{vmatrix} = (-1)^{3+4}\, 3 \begin{vmatrix} 1 & -1 & -2 \\ 2 & 2 & -2 \\ 1 & 1 & 8 \end{vmatrix} \quad \begin{matrix} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - R_1 \end{matrix} \\ &= -3\begin{vmatrix} 1 & -1 & -2 \\ 0 & 4 & 2 \\ 0 & 2 & 10 \end{vmatrix} = -3\begin{vmatrix} 4 & 2 \\ 2 & 10 \end{vmatrix} = -108. \end{aligned}$$

So, $x = \dfrac{\det A_1(b)}{\det A} = \dfrac{-108}{-18} = 6.$
Similarly,

$$y = \frac{\det A_2(b)}{\det A} = \frac{30}{-18} = \frac{-5}{3},$$

$$z = \frac{\det A_3(b)}{\det A} = \frac{-6}{-18} = \frac{1}{3},$$

$$w = \frac{\det A_4(b)}{\det A} = \frac{6}{-18} = \frac{-1}{3}.$$

A FORMULA FOR A^{-1}

Let A be an invertible $n \times n$ matrix.
Let A_{ij} be the submatrix of A formed by deleting row i and column j.

$C_{ij} = (-1)^{i+j} \det A_{ij}$ is called the $(i,j) - cofactor$ of A.

Cofactor expansion about the i th row:

$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

Cofactor expansion about the j th column:

$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

The **adjoint** of A is denoted by **adj**A and

$$\text{adj}A = \begin{bmatrix} C_{11} & C_{12} & . & . & . & C_{1n} \\ C_{21} & C_{22} & . & . & . & C_{2n} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ C_{n1} & C_{n2} & . & . & . & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & . & . & . & C_{n1} \\ C_{12} & C_{22} & . & . & . & C_{n2} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ C_{1n} & C_{2n} & . & . & . & C_{nn} \end{bmatrix}.$$

$$\boxed{A^{-1} = \frac{1}{\det A}\ \text{adj}A} \text{ and } \boxed{\text{adj}A = (\det A)\ A^{-1}}.$$

Homework: Show that $\text{adj}(AB) = (\text{adj}B)(\text{adj}A)$.

Example: Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$. Find $\text{adj}A$ and A^{-1}.

Solution:

$$\begin{aligned}
C_{11} &= (-1)^{1+1} \begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} = 12, \\
C_{12} &= (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 6, \\
C_{13} &= (-1)^{1+3} \begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix} = -16, \\
C_{21} &= (-1)^{2+1} \begin{vmatrix} 2 & -1 \\ -4 & 0 \end{vmatrix} = 4, \\
C_{22} &= (-1)^{2+2} \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 2, \\
C_{23} &= (-1)^{2+3} \begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix} = 16, \\
C_{31} &= (-1)^{3+1} \begin{vmatrix} 2 & -1 \\ 6 & 3 \end{vmatrix} = 12, \\
C_{32} &= (-1)^{3+2} \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = -10, \\
C_{33} &= (-1)^{3+3} \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} = 16.
\end{aligned}$$

$$\text{adj}A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}.$$

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 3 \cdot 12 + 2 \cdot 6 + (-1) \cdot (-16) = 64.$$

Thus,

$$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}A = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} 3/16 & 1/16 & 3/16 \\ 3/32 & 1/32 & -5/32 \\ -1/4 & 1/4 & 1/4 \end{bmatrix}.$$

Homework: First calculate the adj$A$, and then use it to find A^{-1}.

(i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **(ii)** $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 9 \\ -1 & 3 & 0 \end{bmatrix}$ Ans: $A^{-1} = \begin{bmatrix} 3/5 & 0 & -2/5 \\ 1/5 & 0 & 1/5 \\ -11/5 & 1/9 & 1/5 \end{bmatrix}$

Example: Let A be an $n \times n$ invertible matrix. Calculate det(adj A).
Solution:

$$\begin{aligned} \text{adj } A &= (\det A)\, A^{-1} \\ \det(\text{ adj } A) &= \det((\det A) \cdot A^{-1}) \\ &= (\det A)^n \cdot (\det A^{-1}) \\ &= (\det A)^n \cdot \frac{1}{\det A} \\ &= (\det A)^{n-1} \end{aligned}$$

$$\boxed{\det(\text{ adj } A) = (\det A)^{n-1}}$$

Determinants as Area or Volume

Consider a 2×2 matrix $A = \begin{bmatrix} u & v \end{bmatrix}$. Then the area of the parallelogram determined by vectors

$$u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \text{ is } |\det A| = \left|\det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}\right|.$$

Example: Find the area of the parallelogram defined by

$$u = \begin{bmatrix} -3 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

Solution: $A = \begin{bmatrix} u & v \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} \Longrightarrow \text{Area} = |\det \text{A}| = |-15| = 15.$

Example: Find the area of the parallelogram defined by

$$u = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Solution:

$$A = \begin{bmatrix} u & v \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 2 \end{bmatrix} \Longrightarrow \text{Area} = |\det \text{A}| = |4 - 24| = 20.$$

Example: Find the area of the parallelogram with the vertices

$$(0, -2), (6, -1), (3, 2), (-3, 1).$$

Solution: Translate the parallelogram to one having the origin as a vertex. Translate the vertex $(0, -2)$ to the origin.
New vertices: $(0, 0), (6, 1), (3, 4), (-3, 3)$.

$$\text{Area} \quad = \quad \left|\det \begin{bmatrix} -3 & 6 \\ 3 & 1 \end{bmatrix}\right| = |-21| = 21.$$

It does not matter which vertex is translated to the origin. If the vertex $(6, -1)$ is translated to the origin, then the new vertices are
$(-6, -1), (0, 0), (-3, 3), (-9, 2)$.

$$\text{Area} = \left|\det \begin{bmatrix} -3 & -6 \\ 3 & -1 \end{bmatrix}\right| = 21.$$

Example: Find the area of the triangle determined by the vectors $(2,1)$, $(3,5)$, $(7,4)$.
Solution: Translate the triangle to one which has the origin as a vertex. Translate the vertex $(2,1)$ to the origin.
New vertices: $(0,0),(1,4),(5,3)$.

$$\text{Area} \;=\; \frac{1}{2}\left|\det\begin{bmatrix}1 & 5\\ 4 & 3\end{bmatrix}\right| = \frac{1}{2}\,|-17| = \frac{17}{2}.$$

The volume of the parallelepiped determined by the vectors

$$u = \begin{bmatrix}x_1\\ y_1\\ z_1\end{bmatrix},\; v = \begin{bmatrix}x_2\\ y_2\\ z_2\end{bmatrix},\; w = \begin{bmatrix}x_3\\ y_3\\ z_3\end{bmatrix}$$

is

$$\left|\det\begin{bmatrix}u & v & w\end{bmatrix}\right| = \left|\det\begin{bmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{bmatrix}\right|.$$

Example: Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at

$$(1,4,0),\; (-2,-5,2),\; (-1,2,-1).$$

Solution: $A = \begin{bmatrix}1 & -2 & -1\\ 4 & -5 & 2\\ 0 & 2 & -1\end{bmatrix} \Longrightarrow \text{Volume} = |\det A| = |-15| = 15.$

Let $T : R^n \longrightarrow R^n$ be a linear transformation, $S \subset R^n$, $n = 2, 3$.
What is the relation between the area of S and the area of $T(S)$?
What is the relation between the volume of S and the volume of $T(S)$?

Theorem:

- Let $T : R^2 \longrightarrow R^2$ be the linear transformation defined by a 2×2 matrix A.
  If S is a parallelogram in R^2, then

$$\boxed{\{\text{area of } T(S)\} = |\det A| \cdot \{\text{ area of } S\}}$$

- Let $T : R^3 \longrightarrow R^3$ be the linear transformation defined by a 3×3 matrix A. If S is a parallelepipep in R^3, then

$$\boxed{\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{ volume of } S\}}$$

Example: Let S be the parallelogram determined by the vectors

$$u = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } v = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

Let T be the linear transformation defined by

$$T(x) = Ax, \text{ where } A = \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}.$$

Find the area of $T(S)$.

Solution:

$$\{\text{area of } S\} = |\det[u\ v]| = \left|\det \begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix}\right| = 11.$$

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{ area of } S\} = 10 \cdot 11 = 110 \text{ unit}^2$$

$$\begin{aligned} \{\text{area of } T(S)\} &= \left|\det \begin{bmatrix} T(u) & T(v) \end{bmatrix}\right| = \left|\det \begin{bmatrix} 16 & 18 \\ -7 & -1 \end{bmatrix}\right| \\ &= |-16 + 126| = 110 \text{ unit}^2. \end{aligned}$$

Example: Let S be the the parallelepiped with one vertex at the origin and adjacent vertices at

$$u = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}, v = \begin{bmatrix} -2 \\ -5 \\ 2 \end{bmatrix}, \text{ and } w = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}.$$

Let T be the linear transformation defined by

$$T(x) = Ax, \text{ where } A = \begin{bmatrix} 2 & 7 & 3 \\ 0 & -4 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find the volume of $T(S)$.

Solution:

$$\{\text{volume of } S\} = \left| \det \begin{bmatrix} 1 & -2 & -1 \\ 4 & -5 & 2 \\ 0 & 2 & -1 \end{bmatrix} \right| = 15.$$

$$\left\{ \begin{array}{c} \text{volume of} \\ T(S) \end{array} \right\} = |\det A| \cdot \{\text{volume of } S\} = |-8| \cdot 15 = 120 \text{ unit}^3$$

Another way:

$$\begin{aligned} \left\{ \begin{array}{c} \text{volume of} \\ T(S) \end{array} \right\} &= \left| \det \begin{bmatrix} T(u) & T(v) & T(w) \end{bmatrix} \right| \\ &= \left| \det \begin{bmatrix} 30 & -33 & 9 \\ -16 & 22 & -9 \\ 0 & 2 & -1 \end{bmatrix} \right| \\ &= \left| \det \begin{bmatrix} 30 & -15 & 0 \\ -16 & 4 & 0 \\ 0 & 2 & -1 \end{bmatrix} \right| \\ &= |-(120 - 240)| \\ &= 120 \text{ unit}^3. \end{aligned}$$